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Variational Methods in Cavitational Flow

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ABSTRACT

The study of cavitational flow is formulated as a free boundary problem for the Laplace equation in three dimensions. Constant pressure free streamlines are determined by a variational principle for the virtual mass. Steepest descent applied to minimization of the potential energy suggests a natural iteration scheme to calculate the shape of the cavity bounded by the free streamlines. Numerical methods enable one to estimate the drag and the geometry of the flow. Another version of the variational principle plays an important role in plasma physics and the theory of magnetic fusion. Novel stellarator configurations for a thermonuclear reactor have been designed by running large computer codes based on these mathematical ideas.

1. Introduction

We shall be concerned with steady, irrotational flow of an incompressible fluid with a free surface on which the pressure is constant. This is a difficult mathematical problem because the shape of the free boundary must be calculated as part of the solution. Our approach is to apply the principle of minimum virtual mass, in which the answer appears as the solution of a problem in the calculus of variations that is easier to treat both theoretically and numerically. The appearance of potential energy in the analysis may be unconventional in fluid dynamics, but we shall show how this can be used to find the drag, and afterwards we shall present another application of the same method in magnetohydrodynamics where the physics becomes more natural.

2. The Riabouchinsky model

Steady irrotational flow of an incompressible fluid in the plane is governed by a complex potential $\zeta = \phi + i\psi$ whose real and imaginary parts satisfy the Cauchy-Riemann equations

$$\phi_x = \psi_y \, , \; \phi_y = -\psi_x \, .$$

The horizontal and vertical components of the velocity are found from the derivative

$$w = u - iv = d\zeta/dz$$

of ζ as an analytic function of the complex variable $z = x + iy$, and the pressure p is found from Bernoulli's law

$$\frac{q^2}{2} + \frac{p}{\rho} = \text{const.} \, ,$$

where $q = |w|$ is the speed. Thus along any streamline $\psi = 0$ bordering a cavity we arrive at a free boundary condition of the form

$$p = \text{const.} \quad , \quad q = \text{const.}$$

In the Riabouchinsky model of cavitational flow the obstacle consists of two symmetrically located vertical plates joined by a pair of free streamlines. A solution is obtained by mapping the upper half of the ζ-plane conformally onto a semicircle in the hodograph plane slit along a segment of the u-axis. Another interesting method of finding the flow is to study the geometry of the analytic function

$$g(z) = \int \zeta'(z)^2 \, dz = \bar{z} \, ,$$

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which has boundary values indicated at the right along a free streamline where $q = 1$. This function has to do with the forces on the vertical plates and is suggested by the conservation of momentum. The solution of the Riabouchinsky free boundary value problem can be obtained by using the Schwarz-Christoffel formula to calculate the two auxiliary functions

$$\Phi(z) = z + g(z) , \ \Psi(z) = z - g(z) ,$$

which map the flow conformally onto polygonal domains of the complex plane. These classical methods apply primarily to plane flow and play a significant role in the study of supercavitating hydrofoils. However, we do not discuss them in detail here, but turn our attention instead to three-dimensional flows such as occur in the study of torpedoes.

3. The principle of minimum virtual mass

Let us consider a flow of water past a body followed by a cavity Ω in three-dimmensional space. The velocity potential of the flow is a harmonic function ϕ whose normal derivative vanishes at the boundary, and at infinity it has an expansion of the form

$$\phi = x + \frac{ax}{r^3} + \ldots$$

with a coefficient a related to the virtual mass. On the free surface Γ bounding the cavity it satisfies the additional condition

$$(\nabla\phi)^2 = \text{const.}$$

because the pressure is constant there. The shape of the cavity has to be adjusted to meet this nonlinear free boundary condition, and that is what makes the problem hard mathematically.

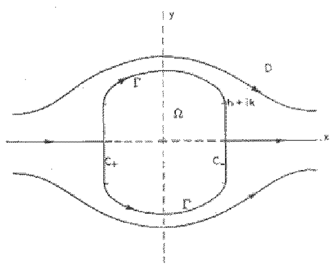

Figure 1: Geometry of the Riabouchinsky model of plane flow around a finite cavity Ω displaying the free stramlines Γ and two vertical fixed boundaries C.

The kinetic energy, or virtual mass,

$$M = \int\int\int (\nabla\phi - \nabla x)^2 \, dV$$

is a Dirichlet integral over the flow region. We also introduce the volume

$$V = \int\int\int dV ,$$

which is an integral over the cavity and the body. An application of the divergence theorem leads to the remarkable formula

$$4\pi a = M + V$$

relating the virtual mass to the coefficient a in the expansion of the velocity potential at infinity. We proceed to discuss variational formulas for a and V under a suitable hypothesis about smoothness at the boundary of the flow. Let us shift each point on the free surface by an infinitesimal distance

δn along its normal n. It is evident that the first order perturbation of the volume is given by the formula

$$\delta V \; = \; -\int\int \delta n \; dS \; ,$$

where integration is performed over the free surface.
Next observe that according to the divergence theorem, the harmonic function ϕ minimizes the Dirichlet integral regardless of conditions at the boundary because its normal derivative vanishes there. It follows that the perturbation of the coefficient a under the shift δn can be computed with ϕ held fixed. Therefore we obtain an equally simply result

$$4\pi\delta a \; = \; -\int\int (\nabla\phi)^2 \delta n \; dS$$

giving the first variation of the virtual mass. This means that a flow satisfying the free boundary condition

$$(\nabla\phi)^2 \; = \; \lambda$$

can be found from the variational principle

$$4\pi\delta a - \lambda\delta V \; = \; 0 \; ,$$

where λ is constant. The result is better stated in the form

$$M - \sigma V \; = \; \text{minimum}$$

of an extremal problem, where $\sigma = \lambda - 1$ is the cavitation parameter.
The theorem we have described, which we refer to as the principle of minimum virtual mass, makes sense only when the cavity is finite so the improper integrals that occur are convergent. This becomes the case for the Riabouchinsky model of cavitational flow. In that model two symmetrically situated plates C are connected by the free surface Γ of a cavity Ω. For plane flow the geometry is illustrated in Fig. 1. Note that the cavitation parameter σ is positive.
A physical interpretation of the variational formula for the virtual mass just states that work equals force times displacement. Details of a mathematical proof are easier to understand for the example of a vortex ring, which will come up again in our discussion of magnetohydrodynamics. Suppose ϕ is a harmonic function in a torus D with a unit period and a vanishing normal derivative, and let ϕ^* be the analogous function in a neighboring torus D^* obtained by a shift δn of the boundary. If M now stands for the Dirichlet integral of ϕ over D and M^* stands for the Dirichlet integral of ϕ^* over D^*, what we want to show is that

$$M^* - M \; = \; \int\int (\nabla\phi)^2 \delta n \; dS \; ,$$

where higher order terms are neglected and the integration is performed over the surface of D. This is true because the Dirichlet integrals of ϕ^* and ϕ over D differ by a term of second order in δn. To see that we apply the divergence theorem to establish the identity

$$\int\int\int \nabla\phi \cdot (\nabla\phi^* - \nabla\phi) \; dV \; = \; \int\int \frac{\partial\phi}{\partial n}(\phi^* - \phi) \; dS - \int\int\int (\phi^* - \phi) \; \Delta\phi \; dV \; = \; 0 \; ,$$

which asserts that because of the natural boundary condition satisfied by ϕ its Dirichlet integral over D is stationary. The desired result now follows because

$$(\nabla\phi + \nabla\phi^* - \nabla\phi)^2 \; = \; (\nabla\phi)^2 + 2\nabla\phi \cdot (\nabla\phi^* - \nabla\phi) + (\nabla\phi^* - \nabla\phi)^2 \; .$$

4. Calculation of the drag

For special shapes of the fixed boundary the principle of minimum virtual mass can be used to calculate the drag. In the Riaboucnsky model suppose that the free surface Γ is a sheath connecting a symmetric pair of circular disks situated in planes perpendicular to the x-axis. Let us magnify the whole configuration by the factor $1+\eta$, where η is a small positive number. For this infinitesimal normal displacement of the boundary of the flow our variational formulas yield the relation

$$\delta M - \sigma \delta V \; = \; \int\int [(\nabla\phi)^2 - \lambda]\delta n \; dS \; .$$

Along the free surface the integrand vanishes, so

$$\delta M - \sigma \delta V \; = \; \eta h \int\int [(\nabla\phi)^2 - \lambda] \; dS \; ,$$

where the integrals are now evaluated only over the disks, and $2h$ is the distance between them. On the other hand, by Bernoulli's law the integral

$$D \; = \; \frac{1}{2} \int\int [1 + \sigma - (\nabla\phi)^2] \; dS$$

over just one disk is the drag. Since V and M have the dimensions of length cubed, and since the shift δn is defined by a magnification, this establishes the remarkable relationship

$$4hD \; = \; 3(\sigma V - M)$$

between the virtual mass and the drag for Riabouchinsky flow past a circular disk.
If the formula we just established for the drag were generalized to the case of two parallel plates of elliptical cross section, then a continuous transition could be made from axially symmetric over to plane flow. It is not hard to see that invariance of the drag under changes of the eccentricity of the ellipse would then lead to a contradiction. We conclude that it is naive to assume that the symmetric model of Riabouchinsky flow always has solutions in three dimensions.
To compute the drag more generally one needs to solve the cavitational flow problem numerically. A systematic scheme to accomplish that is suggested by applying the concept of steepest descent to the principle of minimum virtual mass. This means that to improve on a given approximation of the free surface one should shift it along the normal by an amount δn proportional to the error $(\nabla\phi)^2 - \lambda$ in the free boundary condition. This concept has been implemented in computer codes that run quite succesfully. We shall discuss the matter in more detail when we describe an application to magnetic fusion in the next section.
Another attack on the drag problem has been developed using a stream function ψ which satisfies the partial differential equation

$$\Delta\psi - \frac{\epsilon}{y}\frac{\partial\psi}{\partial y} \; = \; 0$$

with $\epsilon = 0$ for plane flow and $\epsilon = 1$ for axially symmetric flow. For other values of ϵ the boundary value problem still makes sense, and it can be solved in closed form in special cases. That enables one to estimate the drag for a circular disk by interpolation from known values. For a circular disk with an infinite cavity the drag coefficient is found to be $C_D = 0.827$. Similarly, the contraction coefficient for a circular orifice in a plane wall is $C_c = 0.59$, a result differing from the answer obtained in the case of plane flow. Presently of course many more complicated free boundary problems are being investigated computationally by a variety of new methods.

5. Variational principle of magnetohydrodynamics

Another enlightening application of the variational principle plays an important role in the plasma physics of magnetic fusion. The concept of a thermonuclear reactor is based on fusion of hydrogen at very high temperatures to form helium and release neutrons that penetrate a blanket which is heated up to supply power. The hydrogen is ionized to become a plasma so hot that it should not encounter material walls and must be confined by a magnetic field. The charged particles of the plasma tend to follow lines of force in the magnetic field, whose geometry is therefore usually chosen to be a torus. Most of the magnetic lines sweep out nested flux surfaces rather than ergodic regions so that the confinement of the plasma is adequate for fusion in a power plant. There is an analogy here with vortex rings and rotational flow of an incompressible fluid governed by the Euler equations. We shall review the mathematical theory of equilibrium and stability in magnetohydrodynamics that is required to address this problem.

We study the equilibrium and stability of a plasma by solving magnetostatic equations

$$\nabla \cdot \mathbf{B} \;=\; 0, \quad \mathbf{J} \times \mathbf{B} \;=\; \nabla p$$

analogous to the Euler equations, where $\mathbf{B}$ is the magnetic field, $\mathbf{J} = \nabla \times \mathbf{B}$ is the current density, and p is the pressure. In the analogy the velocity of the flow corresponds to the magnetic field and the stagnation enthalpy from Bernoulli's law corresponds to the pressure of the plasma. We shall explain our variational theory in the context of the plasma problem, which will show how islands whose flux surfaces are shaped like vortices can be modeled in practice by current sheets.

Let us rewrite the partial differential equations of magnetostatics in a conservation form

$$\nabla \cdot \mathbf{B} \;=\; 0, \quad \nabla \cdot \mathbf{T} \;=\; 0$$

involving the Maxwell stress tensor $\mathbf{T}$, which is given by

$$T_{jk} \;=\; B_j B_k - \delta_{jk}(B^2/2 + p) \; ,$$

where δ_{jk} is the Kronecker delta. To avoid assuming the existence of partial derivatives, we apply the the divergence theorem and say that $\mathbf{B}$ and p define a weak solution of the equations whenever

$$\int\!\!\int\!\!\int \sum B_k \frac{\partial \psi}{\partial x_k} dV = 0 \; , \qquad \int\!\!\int\!\!\int \sum\sum T_{jk} \frac{\partial \psi_j}{\partial x_k} dV = 0$$

over any volume of integration, where ψ_1, ψ_2, ψ_3 and ψ are arbitrary continuously differentiable functions of compact support. The simplest example of a weak solution is a magnetic field with just one nontrivial component B_1 that is a nondifferentiable function of x_2 and x_3, while the pressure satisfies the condition $B_1^2/2 + p = \text{const}.$

These formulas are comparable to the conservation of mass

$$\sum \frac{\partial u_k}{\partial x_k} \;=\; 0$$

and the conservation of momemtum

$$\sum \frac{\partial}{\partial x_k} \; u_j u_k + \frac{\partial p}{\rho \partial x_j} \;=\; 0$$

for steady flow of an incompressible fluid without viscosity in hydrodynamics. Another application of the divergence theorem establishes that any smooth surface of discontinuity of a weak solution of the magnetostatic equations must be a current sheet in the sense that it is a flux surface of the magnetic field across which $\mathbf{B}$ may have jumps, but $B^2/2 + p$ remains continuous. It is our contention that small magnetic islands can be modeled computationally by such current sheets when a finite difference scheme with adequate numerical viscosity is applied. We use the variational principle of magnetohydrodynamics to show how this concept has been implemented in practice. The analogy with the problem of cavitational flow is evident.

The variational principle of magnetohydrodynamics enables one to study questions of toroidal equilibrium and stability in plasma physics by considering the extremal problem

$$\int\int\int [B^2/2 - p]dV = \text{minimum}$$

for the potential energy subject to appropriate flux constraints on the vector field $\mathbf{B}$, which is supposed to be divergence free. This leads in a natural way to the Clebsch representations

$$\mathbf{B} \;=\; \nabla s \times \nabla \theta$$

of $\mathbf{B}$ in terms of flux functions s and θ and potentials ϕ and ζ. We make a nested surface hypothesis under which s becomes the single-valued toroidal flux and p is a prescribed function of s. The flux function θ is multiple-valued on each torus $s =$ const., with periods related to the rotational transform $\iota = \iota(s)$, which measures how much a magnetic line turns in the poloidal direction during a complete circuit in the toroidal direction. For fully three-dimensional equilibria called stellarators no constraint is imposed on ι in the variational principle and the potential ϕ has no poloidal period, whereas for an axially symmetric tokamak ι is a prescribed function of the toroidal flux s and ϕ acquires a nontrivial poloidal period equal to the net current.

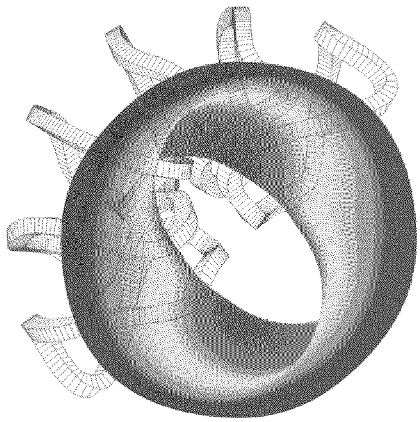

Figure 2: Computational model of fusion plasma in a thermonuclear reactor based on the stellarator concept for magnetic confinement of hot ions and electrons. Twelve moderately twisted modular coils, half of which are plotted, produce a magnetic field whose strength has a desirable symmetry displayed by the color map of the plasma surface.

Straightforward perturbations of the potential energy lead to variational equations that are equivalent to classical magnetostatics. Before a standard integration by parts they furnish a practical definition, essentially in conservation form, of what we mean by a weak solution without derivatives. For a stellarator at zero $\beta = 2\langle p \rangle / B^2$ the result reduces to Dirichlet's principle for harmonic functions, so the configuration is stable. The NSTAB computer code has been written to implement the variational principle of magnetohydrodynamics numerically and perform an analysis of nonlinear stability for positive β. It enables us to capture small magnetic islands in a convincing way when we use a mesh size for s larger than the island width.

In stellarators it is convenient to renormalize the multiple-valued flux function θ and the Clebsch potential ϕ so they become invariant poloidal and toroidal angles. If one then represents the magnetic field strength by a Fourier series of the form

$$\frac{1}{B^2} = \sum B_{mn} \cos(m\theta - [n - \iota m]\phi) \ ,$$

the coefficients B_{mn}, which depend on the toroidal flux s, are known collectively as the magnetic spectrum. By solving the magnetostatic equations for J and taking a divergence we can show that there is a comparable expansion

$$\frac{\mathbf{J}\cdot\mathbf{B}}{B^2} = p' \sum \frac{mB_{mn}}{n - \iota m} \cos(m\theta - [n - \iota m]\phi)$$

for the parallel current, which is analogous to the swirl in fluid dynamics. In three dimensions the small denominators $n - \iota m$ are seen to vanish at a dense set of rational surfaces where $\iota = n/m$. Therefore smooth solutions of the equilibrium problem that have three-dimensional asymmetry do not in general exist, so that one should only try to construct weak solutions. Correspondingly, the only smooth vortex rings in steady flow must be axially symmetric.

The NSTAB code includes a remarkably robust calculation of the coefficients B_{mn}. In the spectral method that has been used it is helpful to filter the Fourier series defining the solution. This is especially true in the case of the parallel current, which requires evaluation of a divergent series. It turns out that many aspects of equilibrium, stability and transport just depend on the magnetic spectrum, together with the profiles of pressure and rotational transform. For tokamaks there is a two-dimensional symmetry such that only the column B_{m0} differs from zero, and similarly for straight two-dimensional stellarators only the diagonal B_{mm} is present. Because of their two-dimensional magnetic symmetry, these equilibria have excellent transport properties. The problem of design is to exploit this theory to arrive at configurations for a fusion reactor that meet a broad range of physics and engineering requirements.

6. Quasiaxially symmetric stellarators

Fast computer codes with a three-dimensional capability are an essential tool for the design of efficient stellarators. Here we shall discuss in some detail a stellarator called the Modular Helias-like Heliac 2 (MHH2) that has quasiaxial symmetry characterized by very small coefficients B_{mn} with $n \neq 0$. This was discovered recently by running the NSTAB code. It has just two field periods and the aspect ratio of the plasma is only 2.5. Transport is almost as good as that in a tokamak because of the comparable symmetry. Moderately twisted modular coils to generate the required magnetic field can be wound on a control surface surrounding the plasma, whose shape brings the total rotational transform into an acceptable range $0.5 \geq \iota \geq 0.4$. Ample space is available for the hardware requirements of a power plant (cf. Fig. 2).

We studied nonlinear stability of the MHH2 stellarator by looking for bifurcated equilibria over the full torus that do not have the helical symmetry of the solution with two field periods. The most dangerous mode we found running the NSTAB code has a complicated ballooning structure that is localized in an outer rim of the plasma. Equilibrium seems to impose a more severe limitation on β for the MHH2 than stability. Only for broad pressure profiles do good magnetic surfaces prevail when β is raised as high as 5%. Thus calculations by the NSTAB code used to invent the MHH2 establish that its equilibrium and stability limits on average β for a relatively broad pressure profile lie somewhere between 4% and 5%. The configuration is sufficiently robust to allow for further optimization. Less painstaking computations of linear or local stability that ignore the difficulties with parallel current at magnetic resonances give misleading predictions about performance. Numerical simulations built around the construction of weak solutions seem to be the best that mathematical theory can contribute to the problem of design in this situation.

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